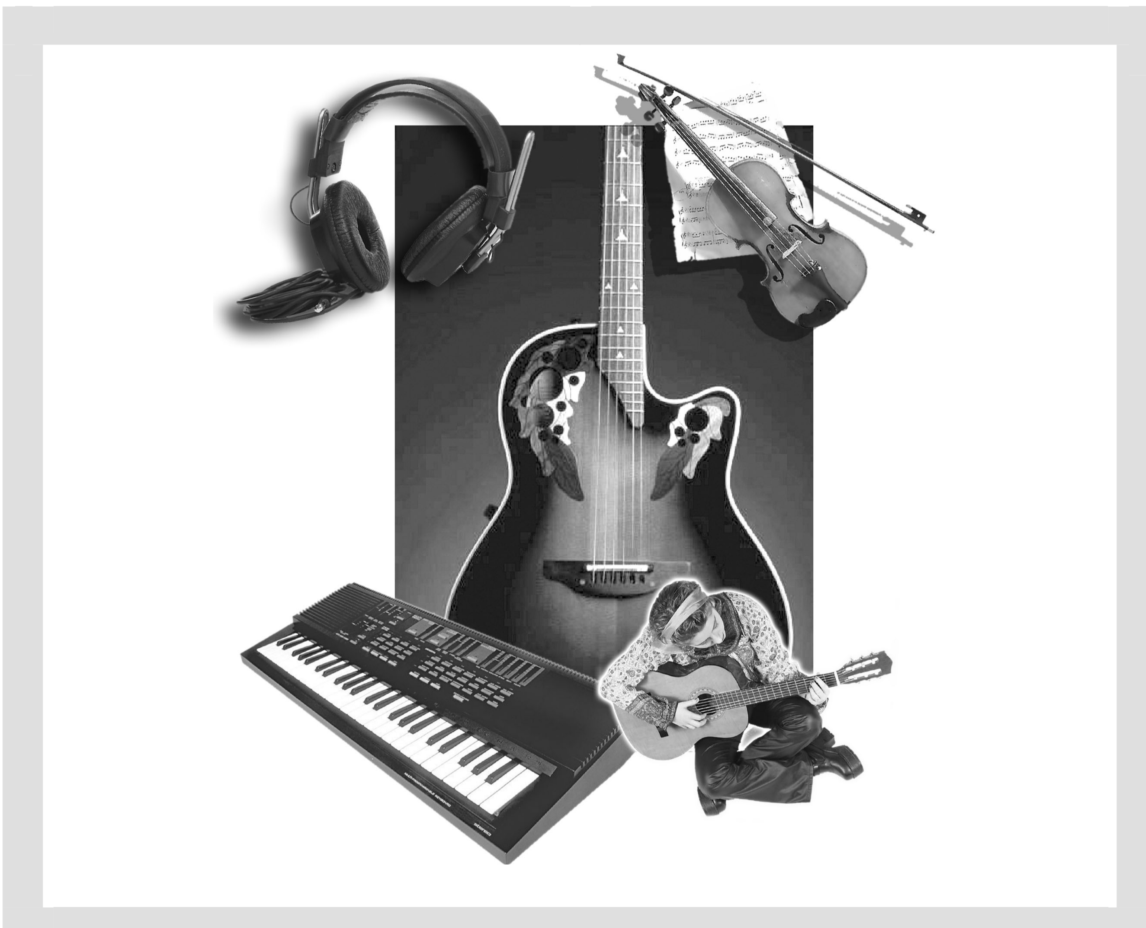
Pure Mathematics 30

Student Project: Mathematics and Music



February 2007

Alberta
EDUCATION

Pure Mathematics 30

Project: Mathematics and Music

Student Task

Mathematics and music are interconnected topics. In her book *Math and Music: Harmonious Connections*, Charity Kahn states, “Music gives beauty and another dimension to mathematics by giving life and emotion to the numbers and patterns.” Mathematical concepts and equations are connected to the designs and shapes of musical instruments, scale intervals and musical compositions, and the various properties of sound and sound production.

This project will allow you to explore several aspects of mathematics related to musical concepts.

Piano Keyboard

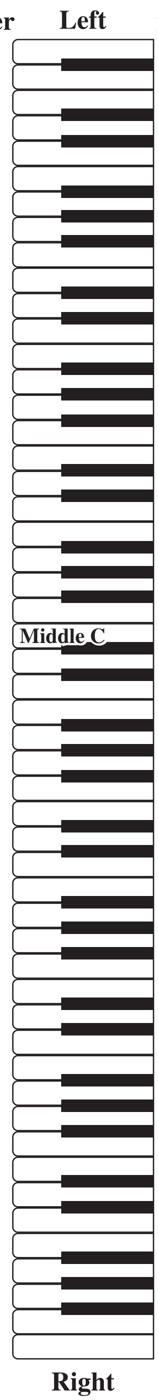
For Part A and Part B of this project, it is important to have some knowledge of the piano keyboard, which is illustrated to the right.

This keyboard has 88 keys of which 36 are black keys and 52 are white keys. Beginning with key 1 (A0) on the left side of the keyboard (the top of the illustration), striking each successive key produces a pitch with a particular frequency that is higher than the pitch produced by striking the previous key by a fixed interval called a semitone. The frequencies increase from left to right. Some examples of the names of the keys are A0, A0#, B0, C1, C1#.

For the purposes of this project, all the black keys will be referred to as sharps (#).

White Keys

Key name	Key number
A0	1
B0	3
C1	4
D1	6
E1	8
F1	9
G1	11
A1	13
B1	15
C2	16
D2	18
E2	20
F2	21
G2	23
A2	25
B2	27
C3	28
D3	30
E3	32
F3	33
G3	35
A3	37
B3	39
C4	40
D4	42
E4	44
F4	45
G4	47
A4	49
B4	51
C5	52
D5	54
E5	56
F5	57
G5	59
A5	61
B5	63
C6	64
D6	66
E6	68
F6	69
G6	71
A6	73
B6	75
C7	76
D7	78
E7	80
F7	81
G7	83
A7	85
B7	87
C8	88

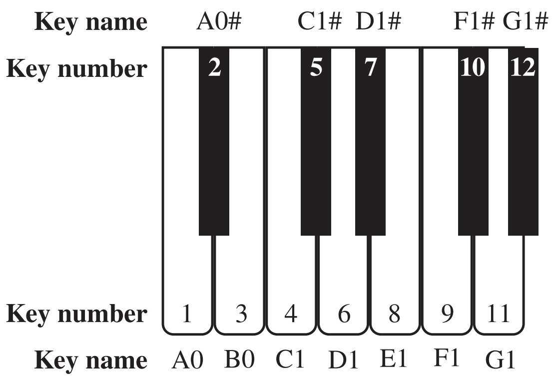


Black Keys

Key number	Key name
2	A0#
5	C1#
7	D1#
10	F1#
12	G1#
14	A1#
17	C2#
19	D2#
22	F2#
24	G2#
26	A2#
29	C3#
31	D3#
34	F3#
36	G3#
38	A3#
41	C4#
43	D4#
46	F4#
48	G4#
50	A4#
53	C5#
55	D5#
58	F5#
60	G5#
62	A5#
65	C6#
67	D6#
70	F6#
72	G6#
74	A6#
77	C7#
79	D7#
82	F7#
84	G7#
86	A7#

Part A

The frequencies of all successive pitches produced by striking the keys on a piano keyboard form a pattern. The diagram on the left shows the first 12 keys of a piano. The table on the right shows the frequency of the pitch produced by each key, to the nearest thousandth of a Hertz (Hz).



Key Name	Key Number	Frequency (Hz)
A0	1	27.500
A0#	2	29.135
B0	3	30.868
C1	4	32.703
C1#	5	34.648
D1	6	36.708
D1#	7	38.891
E1	8	41.203
F1	9	43.654
F1#	10	46.249
G1	11	48.999
G1#	12	51.913

1. The frequencies of the pitches produced by striking the piano keys can be modelled by an exponential regression function.

- Into your calculator, enter the 12 key numbers (1, 2, 3, …, 12) in list L_1 and the corresponding frequencies (27.500, 29.135, …, 51.913) in list L_2. Graph L_2 as a function of L_1, using an appropriate window setting.

- Determine the exponential regression function that models your graph. Write this function in the form $y = ab^x$, with a and b rounded to seven decimal places.

- Given the exponential regression function you determined, what is the rate of increase of the frequency produced by striking any two successive keys?

2. The frequencies of pitches produced by striking the piano keys can also be modelled by a geometric sequence. The model can be determined by using a pair of keys with the same letter and consecutive numbers; for example, A0 and A1, or B1 and B2, or G2# and G3#. Each pair of consecutive keys with the same letter has frequencies with a ratio of 2:1.

In other words, the frequency of A1 (55.000 Hz) is double the frequency of A0 (27.500 Hz), the frequency of A2 (110.000 Hz) is double the frequency of A1 (55.000 Hz), and so on.

- For a geometric sequence with a first term of 27.500 (frequency of key 1, A0) and a thirteenth term of 55.000 (frequency of key 13, A1), what is the common ratio, r, as an exact value and as an approximate value to seven decimal places?
- Write the general term formula, $t_n = ar^{n-1}$, that describes the geometric pattern of the frequencies of successive pitches produced by striking the keys of the piano.

3. State which of the two equation models most appropriately describes the pattern of frequency increase of pitches produced by a piano. Justify your choice with an explanation.

4. **a.** Using your geometric sequence formula from question 2, determine the frequency corresponding to the pitch represented by

- key 35 (G3)
- key 52 (C5)

b. What is the range of frequencies that striking the keys on a piano can produce?

5. • Rewrite the equation $t_n = ar^{n-1}$ to solve for n. In the rewritten equation, substitute $a = 27.5$ and $r = \sqrt[12]{2}$.

6. On a particular piano that is slightly out of tune, the playing of a single key produces a frequency of 783.5 Hz. Use the equation from question 5 to determine which key number best corresponds to a pitch of this frequency.

Part B

When tuning a musical instrument such as a piano, a violin, or a guitar, a tuning fork or an electronic tuning device is used to produce a pure pitch at a specific frequency.

The tuning fork used most often is for the key A4 ($f = 440$ Hz) and is referred to as an A440 tuning fork. A tuning fork produces a sound wave whose pressure variations can be modelled by a sinusoidal function of the form

$$P = a\sin(2\pi f t),$$

where P = pressure variation of the sound wave in pascals (Pa)
a = amplitude of the sound wave in pascals
f = frequency of the sound wave in Hertz (Hz)
t = time in seconds

1. A violin is being tuned with an A440 tuning fork ($f = 440$ Hz). The sound wave that is produced by the tuning fork has an amplitude of 0.10 Pa. This sound wave can be represented by the function

$$P = 0.10\sin(880\pi t).$$

- Using your graphing calculator in radian mode and with window settings of

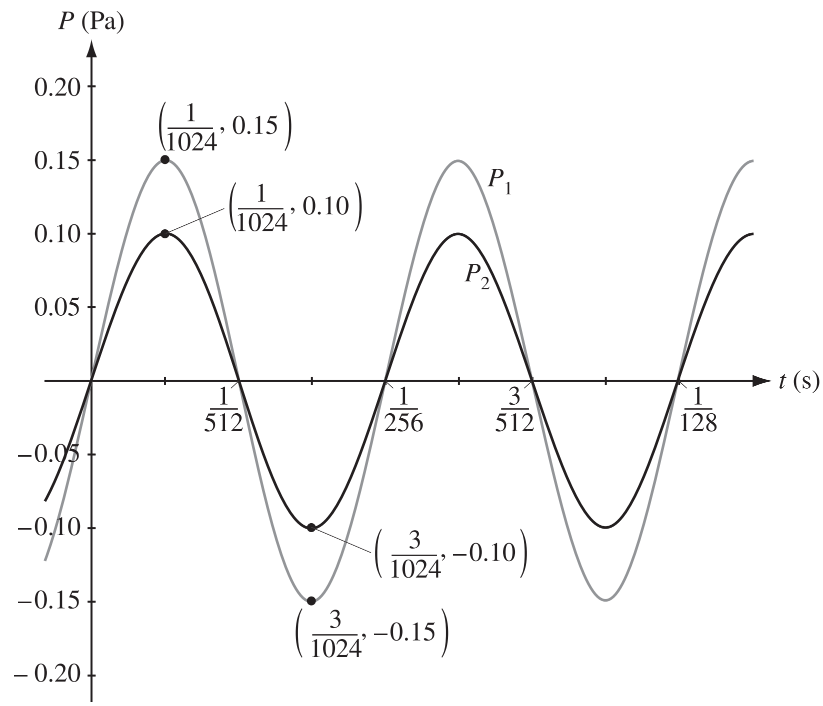
  x: [0, 0.005, 0.0005] y: [– 0.15, 0.15, 0.005],

  sketch the graph of this function. Label each axis and indicate the scale used.

- State the domain, range, and period for the sound wave represented by the function $P = 0.10\sin(880\pi t)$.

- The frequency of the sound wave, f, is defined in terms of cycles/second or Hertz, and the period of the sound wave, T, is defined in terms of seconds/cycle. Describe the identity that expresses the relationship between frequency and period. Verify this identity by using the frequency and period for the sound wave represented by $P = 0.10\sin(880\pi t)$.

2. Another tuning fork with a different frequency is being used to tune a particular pitch (key) on a piano. The tuning fork is struck twice, with a different force each time. A graph representing each of the sound waves produced is shown on the coordinate plane below.

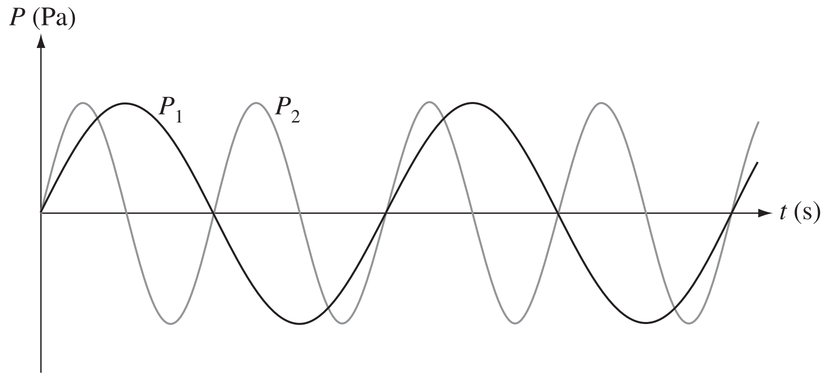


- Explain the similarities and differences between the properties of the two sound waves.

- Write an equation describing the graph of P_1 in the form $P = a\sin(2\pi f t)$.

- Using the equation from Part A, question 5, determine the number of the key that was likely being tuned.

3. Two different tuning forks were each struck once. A graph representing each of the sound waves produced is shown on the coordinate plane below.

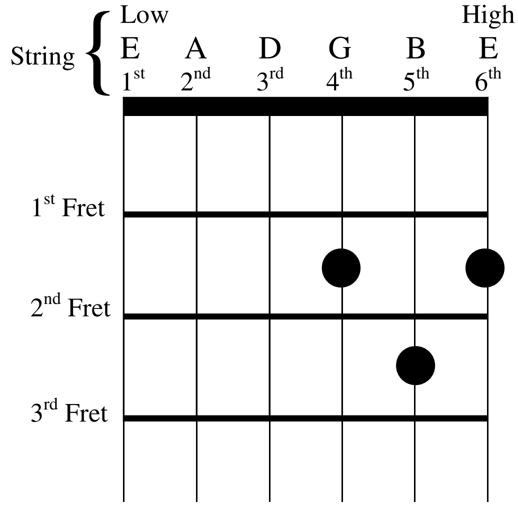


• Explain how the properties of the two sound waves can be used to identify a possible pair of keys being tuned.

Part C

Background

A right-handed guitarist produces a chord by first placing the fingers of his left hand in particular positions along the neck of a guitar. He then finger-picks or strums the appropriate strings with the fingers and thumb of his right hand to play the chord.



A guitar player wants to play a particular D-chord by using the fingers of his left hand to hold down the 4th, 5th, and 6th strings, as shown in the diagram at right. The strings need to be held down between specific frets.* This can be accomplished using two fingers or three fingers.

Individual guitar players prefer certain two- or three-finger arrangements. However, for this project, you are to consider all possible finger arrangements that meet the conditions described in questions.

Note: The four fingers of the left hand are assigned numbers as follows.
finger 1 = index finger
finger 2 = middle finger
finger 3 = ring finger
finger 4 = pinky finger

* Ridges set across the neck of the guitar.

Positioning the Left Hand

1. A guitarist using two fingers to play this particular D-chord could use one finger as a "barre"* between the 1^{st} fret and the 2^{nd} fret to hold down the 4^{th}, 5^{th}, and 6^{th} strings altogether. Another finger would be used between the 2^{nd} fret and 3^{rd} fret to hold down just the 5^{th} string. In this case, the 5^{th} string is held down twice; once with the barre finger and once with the other finger.

One example of how two fingers can be positioned to play the D-chord is shown to the right.

In order to play the D-chord with two fingers, a guitarist must use

– either finger 1 or finger 2 as the barre

– a finger that is higher in number than the two barre fingers to hold down the 5^{th} string

- How many different arrangements of two fingers can the guitarist use to play the D-chord?

* The finger placed on the neck of the guitar so that it holds down 2 or more strings.

2. When playing the D-chord using three fingers, a guitarist uses a different finger on each string. One example of how three fingers can be positioned to play the D-chord is shown to the right.

In order to play the D-chord with three fingers, a guitarist must use

– any of fingers 1, 2, or 3 to hold down the 4^{th} and 6^{th} strings in either order

– a finger that is higher in number (either 3 or 4) than the two fingers used to hold down the 4^{th} and 6^{th} strings to hold down the 5^{th} string

- How many different arrangements of three fingers can the guitarist use to play the D-chord?

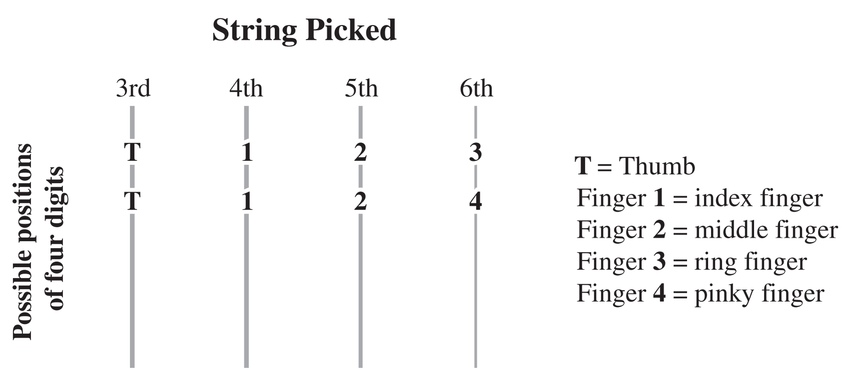
Positioning the Right Hand

3. Once the left hand is positioned, the guitarist can play the D-chord by finger-picking the 3rd, 4th, 5th, and 6th strings.

To finger-pick the strings, he can use the thumb and fingers on his right hand. The conditions that apply are:

– the 3rd string is picked by either the thumb or finger 1
– the 4th string is picked by either finger 1 or finger 2
– the 5th string is picked by either finger 2 or finger 3
– the 6th string is picked by either finger 3 or finger 4
– once a finger is used to pick a string, it cannot be used again

• Two of the choices that the guitarist can use to pick the strings are shown below. List the remaining three choices and explain why the number of choices is ${}_5C_4$.



• Using his thumb and/or fingers as determined above, the guitarist can now play the 4 guitar strings (3rd, 4th, 5th, and 6th) in any order. What is the total number of different orders that he can finger-pick the 4 guitar strings? Show how you obtained your solution.

4. The guitarist has decided that he will finger-pick the 4 guitar strings for this particular D-chord either in ascending order (3rd, 4th, 5th, 6th) or descending order (6th, 5th, 4th, 3rd). Determine the total number of arrangements that the guitarist can use to position the fingers of his left hand **and** pick the strings with his right hand.

Part D

1. Investigate the mathematics related to tension and string length of a guitar or violin in the production of sound frequencies. The following list of web sites may help in your research.
www.noyceguitars.com/Technotes/Articles/T3.html
www.playtheguitar.com/
www.csm.astate.edu/music.html

2. When two sounds of slightly different frequencies are played at the same time (for example, when a string of a violin or a guitar is tuned with a tuning fork), variations of loudness occur at regular intervals. The periodic changes in loudness are called "beats."

As part of the development of the trigonometric function that describes the beat produced, the following identity is used

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \text{ where } A \neq B.$$

Complete a proof of the identity. As a first step on the left side, rewrite $\cos A$ as $\cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right)$ and $\cos B$ as $\cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right)$.

3. Listen to algorithmic music patterns determined by various mathematical concepts (permutations, trigonometric functions, Pascal's triangle, etc.) and explain which patterns appealed to you the most and least.

OR

Design your own musical algorithm based on another mathematical principle or concept (you may even want to present it to the class). The following web site could be used www.geocities.com/Vienna/9349/

Note: Web-site addresses sometimes change. If the web sites above are not available, use a search engine and type in key words such as "Frequency and String Tension" or "Mathematics and Music."

Credits

Page 1: Quotation by Charity Vaughan Kahn. Found on http://charitykahn.com/musimatics/ as quoted from the book, *Math and Music: Harmonious Connections*, by Charity Vaughan Kahn and Trudi Hammel Garland. Dale Seymour Publications, © February 1995. Used with permission.